

MODULE-3-PROBABILITY DISTRIBUTIONS

Introduction

The three main components of statistics are collection of data, analysis of data and inference. The scope of statistics is very vast and finds its applications in almost all spheres. Statistical methods guides us in any scientific enquiry and leans heavily on probability theory.

Basic Definitions and concepts

Experiment is a process by which measurement or observation is obtained.

Random experiment : An experiment which does not result in the same out come when performed under the same conditions.

Eg : 1. Tossing of a fair coin.

2. Throwing of a fair die.

Sample space : The set of all possible outcomes of a random experiment is called a sample space usually denoted by S.

Eg 1. $S = \{H,T\}$ is the sample space in tossing of a single coin.

2. $S = \{HH,TH,HT,TT\}$ is the sample space obtained in tossing of two coins.

3. $S = \{1,2,3,4,5,6\}$ in throwing of a die.

Event : Any event is a subset of an appropriate sample space.

Exhaustive event : An event consisting of all the various possibilities is called an exhaustive event.

Mutually exclusive events : Events are said to be mutually exclusive if and only if one of them can take place at a time, in other words happening of one event prevent the simultaneous happening of the others.

Eg: In the tossing of a coin, either heads or tails may turn up but not both.

Independent events: Two or more events are said to be independent if the happening or non happening of one event does not prevent the happening or non happening of the others.

Eg: 1. When two coins are tossed the event of getting head is an independent event as both the coins can turn out heads.

Probability is the measure of the chance with which we can expect the event to occur. If an event A can occur in m different ways out of number of n possible ways, all of which are equally likely then the probability of the event A denoted by $P(A)$ is given by $P(A) = \frac{m}{n}$.

Note: 1) Since $0 \leq m \leq n \Rightarrow P(A)$ is non negative and $0 \leq P(A) \leq 1$

2) If $P(A) = 0$ then A is called a null event.

3) If $P(A) = 1$ then A is called a sure event.

4) $P(A) + P(\bar{A}) = 1$, the set of all unfavourable events is denoted by $\bar{A}$.

Addition theorem of probability

The probability of the happening of one or the other mutually exclusive events is equal to the sum of the probabilities of the two events.

That is, if A, B are two mutually exclusive events then,

$$P(A \text{ or } B) = P(A) + P(B)$$

Product theorem of probability

If a compound event is made up of a number of independent events, the probability of the happening of the compound event is equal to the product of the probabilities of the independent events.

That is, if A, B are independent events then,

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

RANDOM VARIABLE & PROBABILITY DISTRIBUTION

Introduction

When we consider an experiment of tossing two coins. The sample points or outcomes are qualitative and are described by their attributes though the probabilities attached to each of these outcomes are quantitative. We now associate to each sample point a real number called the **random variable**.

Random variable

Definition: Let S be a sample space then $X : S \rightarrow R$ defined over the elements of S is called a **random variable**. A random variable is a function that assigns a real number to every sample point in the sample space of a random experiment. Random variables are usually denoted by X, Y, Z ... and it may be noted that different random variables may be associated with the sample space S. The set of all real numbers of a random variable X is called the range of X.

Eg:

1. While tossing a coin, suppose that the value 1 is associated for the outcome 'head' and 0 for the outcome tail. We have the sample space $S = \{H, T\}$ and if X is the random variable, then $X(H) = 1$ & $X(T) = 0$.
Range of X = $\{0, 1\}$

2. Suppose a coin is tossed twice, we shall associate two different random variables X, Y as follows, where we have the sample space
 $S = \{HH, HT, TH, TT\}$
X = Number of 'heads' in the outcome.
The association of the elements in S to X is as follows.

Out come	HH	HT	TH	TT
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Random variable X	2	1	1	0
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Range of $X = \{0,1,2\}$

Suppose $Y =$ Number of 'tails' in the outcome, we have

Out come	HH	HT	TH	TT
Random variable Y	0	1	1	2

Range of $Y = \{0,1,2\}$

Discrete and Continuous random variables

If a random variable takes finite or countably infinite number of values then it is called a discrete random variable.

Discrete random variable will have finite or countably infinite range.

Eg: 1. Tossing of a coin and observing the outcome.

2. Throwing a 'die' and observing the numbers on the face.

If a random variable takes non countable number of values then it is called a non discrete or continuous random variable.

Eg:

1. Weight of articles

2. Length of nails produced by machine.

Generally counting problems correspond to discrete random variables and measuring problems lead to continuous random variables.

Probability function and Discrete probability distribution

If for each value x_i of a discrete random variable X , we assign a real number $p(x_i)$ such that

i) $p(x_i) \geq 0$ ii) $\sum_i p(x_i) = 1$ then the function $p(x)$ is called a probability function.

If the probability that X take the values x_i is p_i , then $P(X = x_i) = p_i$ or $p(x_i)$.

The set of values $[x_i, p(x_i)]$ is called a discrete (finite) probability distribution of the discrete random variable X . The function $P(X)$ is called the probability density function (p.d.f) or the probability mass function (p.m.f)

The distribution function $f(x)$ defined by $f(x) = P(X \leq x) = \sum_{i=1}^n P(x_i)$, x being an integer, is called the cumulative distribution function (c.d.f).

The mean and variance of the discrete probability distribution are defined as follows

Mean $\mu = \sum_i x_i \cdot p(x_i)$

Variance $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = \sum_i x_i^2 \cdot p(x_i) - [\sum_i x_i \cdot p(x_i)]^2$$

$$V = \sum x_i^2 \cdot p(x_i) - \mu^2$$

Standard deviation $\sigma = \sqrt{V}$

Problems

1. A coin is tossed twice. A random variable X represent the number of heads turning up.

Find the discrete probability distribution for X . Also find its mean and variance.

Sol: Here $S = \{HH, HT, TH, TT\}$

$$X = \{0, 1, 2\}$$

$$\text{Now, } P(HH) = \frac{1}{4} \quad P(HT) = \frac{1}{4} \quad P(TH) = \frac{1}{4} \quad P(TT) = \frac{1}{4}$$

$$P(X = 0, \text{ i.e., no head turning up}) = P(TT) = \frac{1}{4}$$

$$P(X = 1, \text{ i.e., one head turning up}) = P(HT \cup TH) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(X = 2, \text{ i.e., two heads turning up}) = P(HH) = \frac{1}{4}$$

The discrete probability distribution for X is as follows

$X=x_i$	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

We observe that, $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.

$$\text{Mean, } \mu = \sum_i x_i \cdot p(x_i) = (0) \cdot \left(\frac{1}{4}\right) + (1) \cdot \left(\frac{1}{2}\right) + (2) \cdot \left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{2}$$

$$\mu = 1$$

$$\text{Variance, } V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$$V = (0 - 1)^2 \cdot \frac{1}{4} + (1 - 1)^2 \cdot \frac{1}{2} + (2 - 1)^2 \cdot \frac{1}{4}$$

$$V = \frac{1}{4} + 0 + \frac{1}{4}$$

$$V = \frac{1}{2}$$

2. A random experiment of tossing a 'die' twice is performed. Random variable X on this sample space is defined to be the sum of the two numbers turning up on the toss. Find the

discrete probability distribution for the random variable X and compute the corresponding mean and standard deviation.

Sol: Here $S = \{(x, y) \text{ where } x = 1, 2, \dots, 6 ; y = 1, 2, \dots, 6\}$

i.e., $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), \dots, (6,5), (6,6)\}$

Number of elements in $S=36$.

The set of values of the random variable X defined as sum of two numbers on the face of the 'die' are $\{2,3,4,5, \dots, 10,11,12\}$

Now, $p(x_1) = \frac{n(E)}{n(S)} = \frac{1}{36} ; p(x_2) = \frac{2}{36} ; \dots$

The discrete probability distribution for X is as follows

X = x_i	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Here $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.

Mean $\mu = \sum_i x_i \cdot p(x_i)$

$$\mu = (2) \cdot \left(\frac{1}{36}\right) + (3) \cdot \left(\frac{2}{36}\right) + \dots + (11) \cdot \left(\frac{2}{36}\right) + (12) \cdot \left(\frac{1}{36}\right)$$

$$\mu = \frac{252}{36} = 7$$

Variance, $V = \sum_i (x_i - \mu)^2 \cdot p(x_i)$

$$V = (2 - 7)^2 \cdot \frac{1}{36} + (3 - 7)^2 \cdot \frac{2}{36} + \dots + (12 - 7)^2 \cdot \frac{1}{36}$$

$$V = \frac{210}{36} = \frac{35}{6}$$

Std. deviation , $\sigma = \sqrt{V}$

$$\sigma = \sqrt{\frac{35}{6}} = 2.42$$

3. Show that the following distribution represents a discrete probability distribution. Find

the mean and variance

x	10	20	30	40
p(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Sol: We observe that $p(x) > 0$ and $\sum p(x) = 1$.

Mean $\mu = \sum x \cdot p(x)$

$$\mu = (10) \cdot \frac{1}{8} + (20) \cdot \frac{3}{8} + (30) \cdot \frac{3}{8} + (40) \cdot \frac{1}{8}$$

$$\mu = \frac{10+60+90+40}{8} = \frac{200}{8}$$

$$\mu = 25$$

Variance, $V = \sum (x - \mu)^2 \cdot p(x)$

$$V = (10 - 25)^2 \cdot \frac{1}{8} + (20 - 25)^2 \cdot \frac{3}{8} + (30 - 25)^2 \cdot \frac{3}{8} + (40 - 25)^2 \cdot \frac{1}{8}$$

$$V = \frac{600}{8} = 75$$

4. Find the value of k such that the following distribution represents a finite probability distribution. Hence find its mean and standard deviation. Also find $p(x \leq 1)$, $p(x > 1)$ and $p(-1 < x \leq 2)$

x	-3	-2	-1	0	1	2	3
p(x)	k	2k	3k	4k	3k	2k	k

Sol: We must have $p(x) \geq 0$ for all x and $\sum p(x) = 1$. The first condition is satisfied if $k \geq 0$. Since $\sum p(x) = 1$

$$k + 2k + 3k + 4k + 3k + 2k + k = 1$$

$$16k = 1$$

$$\therefore k = \frac{1}{16}$$

The discrete probability distribution is as follows

x	-3	-2	-1	0	1	2	3
p(x)	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

Mean $\mu = \sum x \cdot p(x)$

$$\mu = (-3) \cdot \frac{1}{16} + (-2) \cdot \frac{2}{16} + (-1) \cdot \frac{3}{16} + (0) \cdot \frac{4}{16} + (1) \cdot \frac{3}{16} + (2) \cdot \frac{2}{16} + (3) \cdot \frac{1}{16}$$

$$\mu = 0$$

Variance, $V = \sum (x - \mu)^2 \cdot p(x)$

$$V = (-3 - 0)^2 \cdot \frac{1}{16} + (-2 - 0)^2 \cdot \frac{2}{16} + (-1 - 0)^2 \cdot \frac{3}{16} + (0 - 0)^2 \cdot \frac{4}{16} + (1 - 0)^2 \cdot \frac{3}{16} + (2 - 0)^2 \cdot \frac{2}{16} + (3 - 0)^2 \cdot \frac{1}{16}$$

$$V = \frac{1}{16} (9 + 8 + 3 + 0 + 3 + 8 + 9) = \frac{40}{16} = \frac{5}{2}$$

Standard deviation $\sigma = \sqrt{V}$

$$\sigma = \sqrt{\frac{5}{2}} = 1.58$$

Now, $p(x \leq 1) = p(-3) + p(-2) + p(-1) + p(0) + p(1)$

$$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16} + \frac{4}{16} + \frac{3}{16}$$

$$p(x \leq 1) = \frac{13}{16}$$

$$p(x > 1) = p(2) + p(3)$$

$$= \frac{2}{16} + \frac{1}{16}$$

$$p(x > 1) = \frac{3}{16}$$

$$p(-1 < x \leq 2) = p(0) + p(1) + p(2)$$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16}$$

$$p(-1 < x \leq 2) = \frac{9}{16}$$

5. A random variable X has the following probability function for various values of x.

x	0	1	2	3	4	5	6	7
P(x)	0	k	2k	2k	3k	k ²	2k ²	7k ² + k

i) Find k ii) Evaluate $P(x < 6)$, $P(x \geq 6)$ and $P(3 < x \leq 6)$. Also find the probability distribution and the distribution function of X.

Sol: We must have $P(x) \geq 0$ and $\sum P(x) = 1$.

The first condition is satisfied for $k \geq 0$ and we have to find k such that

$$\sum P(x) = 1$$

$$\text{i.e., } 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$$

$$10k^2 + 9k - 1 = 0$$

$$(10k - 1)(k + 1) = 0$$

$$k = \frac{1}{10} \quad \text{and} \quad k = -1 \text{ (Neglect)}$$

Hence we have the following table

x	0	1	2	3	4	5	6	7
P(x)	0	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{17}{100}$

$$\text{Now, } P(x < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$

$$= 0 + \frac{1}{10} + \frac{1}{5} + \frac{1}{5} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(x < 6) = 0.81$$

$$P(x \geq 6) = P(6) + P(7)$$

$$P(x \geq 6) = \frac{1}{50} + \frac{17}{100} = \frac{19}{100}$$

$$P(x \geq 6) = 0.19$$

$$P(3 < x \leq 6) = P(4) + P(5) + P(6)$$

$$= \frac{3}{10} + \frac{1}{100} + \frac{1}{50} = \frac{33}{100}$$

$$P(3 < x \leq 6) = 0.33$$

The probability distribution is as follows

x	0	1	2	3	4	5	6	7
P(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

The distribution function of X is $f(x) = P(X \leq x) = \sum_{i=1}^n p(x_i)$ is called cumulative distribution function and same is as follows

x	0	1	2	3	4	5	6	7
f(x)	0	0+0.1 =0.1	0.1+0.2 =0.3	0.3+0.2 =0.5	0.5+0.3 =0.8	0.8+0.01 =0.81	0.81+0.02=0.83	0.83+0.17=1

6. A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such hat $P(X = 0) = P(X < 0)$ and $P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$.Find the probability distribution.

Sol: Let the distribution be as follows

X	-3	-2	-1	0	1	2	3
P(X)	p_1	p_2	p_3	p_4	p_5	p_6	p_7

By data $P(X = 0) = P(X < 0)$

$$\Rightarrow P(X = 0) = P(X = -1) + P(X = -2) + P(X = -3)$$

$$\text{i.e., } p_4 = p_3 + p_2 + p_1 \quad \text{--- (1)}$$

Also we have

$$P(X = -3) = P(X = -2) = P(X = -1) = P(X = 1) = P(X = 2) = P(X = 3)$$

$$\text{i.e., } p_1 = p_2 = p_3 = p_5 = p_6 = p_7 \quad \text{--- (2)}$$

Further we must have

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 = 1 \quad \text{--- (3)}$$

Using (2) in (1), we get $p_4 = p_1 + p_1 + p_1$

$$\mathbf{p_4 = 3p_1}$$

Using (2) in (3), we get $p_1 + p_1 + p_1 + p_4 + p_1 + p_1 + p_1 = 1$

$$6p_1 + p_4 = 1$$

$$6p_1 + 3p_1 = 1 \quad (\text{since } p_4 = 3p_1)$$

$$9p_1 = 1$$

$$\mathbf{p_1 = \frac{1}{9}}$$

$$\text{Hence } \mathbf{p_4 = \frac{3}{9}}$$

$$\text{Thus , } p_2 = p_3 = p_5 = p_6 = p_7 = \frac{1}{9}$$

Thus the probability distribution is as follows

X	-3	-2	-1	0	1	2	3
P(X)	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

7. The p.d.f. of variate X is given by the following table.

x	0	1	2	3	4	5	6
P(x)	K	3K	5K	7K	9K	11K	13K

For what value of K, this represents a valid probability distribution? Also find $P(x \geq 5)$ and $P(3 < x \leq 6)$.

Sol: The probability distribution is valid if $P(x) \geq 0$ and $\sum P(x) = 1$

Hence we must have $K \geq 0$ and $K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$$49K = 1 \Rightarrow K = \frac{1}{49}$$

Also, $P(x \geq 5) = P(5) + P(6)$

$$P(x \geq 5) = 11K + 13K = 24K$$

$$P(x \geq 5) = \frac{24}{49}$$

Now, $P(3 < x \leq 6) = P(4) + P(5) + P(6)$

$$P(3 < x \leq 6) = 9K + 11K + 13K$$

$$P(3 < x \leq 6) = 33K$$

$$P(3 < x \leq 6) = \frac{33}{49}$$

Binomial Distribution

If **p** is the probability of success and **q** is the probability of failure, the probability of **x** success out of **n** trials is given by $P(x) = {}^nC_x p^x q^{n-x}$

Mean and Standard deviation of the Binomial Distribution

$$\text{Mean, } \mu = np$$

$$\text{Variance, } V = npq$$

$$\text{Std. Deviation, } \sigma = \sqrt{npq}$$

Problems

1. Find the binomial probability which has mean 2 and variance $\frac{4}{3}$.

Sol: Given Mean, $\mu = 2$ and Variance, $V = \frac{4}{3}$

For binomial distribution, $\mu = np$

$$np = 2$$

$$npq = V$$

$$2q = \frac{4}{3}$$

$$q = \frac{2}{3}$$

$$\text{Since } p = 1 - q = 1 - \frac{2}{3}$$

$$p = \frac{1}{3}$$

$$\text{Since } np = 2$$

$$\frac{n}{3} = 2$$

$$n = 6$$

The binomial probability function is $P(x) = {}^nC_x p^x q^{n-x}$ becomes

$$P(x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

The distribution of probability is as follows

x	0	1	2	3	4	5	6
P(x)	$\left(\frac{2}{3}\right)^6$	${}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5$	${}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4$	${}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3$	${}^6C_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2$	${}^6C_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1$	$\left(\frac{1}{3}\right)^6$

2. When a coin is tossed 4 times, find the probability of getting i) exactly one head
ii) atmost 3 heads iii) atleast two heads.

$$\text{Sol: Given } n = 4, p = P(H) = \frac{1}{2}, q = P(T) = \frac{1}{2} \quad (\text{since } p + q = 1)$$

From binomial distribution, the probability of x successes out of n trials is given by

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$P(x) = {}^4C_x \left(\frac{1}{2}\right)^4$$

$$P(x) = \frac{1}{16} \cdot {}^4C_x$$

$$\text{i) } P(1 \text{ head}) = P(x = 1) = \frac{1}{16} \cdot {}^4C_1 = \frac{1}{16}(4)$$

$$P(x = 1) = \frac{1}{4}$$

$$P(x = 1) = 0.25$$

$$\text{ii) } P(\text{atmost 3 heads}) = P(x \leq 3)$$

$$P(x \leq 3) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$$

$$P(x \leq 3) = \frac{1}{16} \cdot {}^4C_0 + \frac{1}{16} \cdot {}^4C_1 + \frac{1}{16} \cdot {}^4C_2 + \frac{1}{16} \cdot {}^4C_3$$

$$P(x \leq 3) = \frac{1}{16} [1 + 4 + 6 + 4]$$

$$P(x \leq 3) = \frac{15}{16}$$

$$P(x \leq 3) = 0.9375$$

$$\text{iii) } P(\text{atleast 2 heads}) = P(x \geq 2)$$

$$P(x \geq 2) = 1 - P(x < 3)$$

$$P(x \geq 2) = 1 - [P(x = 0) + P(x = 1)]$$

$$P(x \geq 2) = 1 - \left[\frac{1}{16} \cdot {}^4C_0 + \frac{1}{16} \cdot {}^4C_1 \right]$$

$$P(x \geq 2) = 1 - \frac{1}{16} [1 + 4]$$

$$P(x \geq 2) = 1 - \frac{5}{16}$$

$$P(x \geq 2) = \frac{11}{16}$$

$$P(x \geq 2) = 0.6875$$

3. In a consignment of electric lamps 5% are defective. If a random sample of 8 lamps are inspected. What is the probability that one or more lamps are defective?

$$\text{Sol: Given } n = 8, \text{ Probability of defective lamp } = p = 5\% = \frac{5}{100} = 0.05$$

$$\therefore q = 1 - p = 1 - 0.05$$

$$q = 0.95$$

From binomial distribution,

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^8C_x (0.05)^x (0.95)^{8-x}$$

Probability that one or more lamps are defective

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - {}^8C_0(0.05)^0(0.95)^{8-0}$$

$$P(x \geq 1) = 0.3366.$$

Thus the required probability is 0.3366

4. The probability that a person aged 60 years will live upto 70 is 0.65. What is the probability that out of 10 person aged 60 atleast 7 of them will live upto 70.

Sol: Given $n = 10$, $p = 0.65$

$$\therefore q = 1 - p = 1 - 0.65$$

$$q = 0.35$$

Let x denote the number of person aged 60 years living upto 70 years.

From binomial distribution,

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^{10}C_x (0.65)^x (0.35)^{10-x}$$

Probability that out of 10 person aged 60 atleast 7 of them will live upto 70

$$P(x \geq 7) = P(7) + P(8) + P(9) + P(10)$$

$$P(x \geq 7) = {}^{10}C_7 (0.65)^7 (0.35)^3 + {}^{10}C_8 (0.65)^8 (0.35)^2 + {}^{10}C_9 (0.65)^9 (0.35)^1 + {}^{10}C_{10} (0.65)^{10} (0.35)^0$$

$$P(x \geq 7) = 0.5138$$

5. The number of telephone lines busy at an instant of time is a binomial variate with probability 0.1 that a line is busy. If 10 such lines are chosen at random, what is the probability that i) no line is busy ii) all lines are busy iii) atleast one line is busy iv) atleast 2 lines are busy.

Sol: Given $n = 10$, $p = 0.1$

$$\therefore q = 1 - p = 1 - 0.1$$

$$q = 0.9$$

Let x denote the number of telephone lines busy.

From binomial distribution,

$$P(x) = {}^nC_x p^x q^{n-x}$$

$$P(x) = {}^{10}C_x (0.1)^x (0.9)^{10-x}$$

i) Probability that no line is busy

$$P(x = 0) = {}^{10}C_0 (0.1)^0 (0.9)^{10-0} = (0.9)^{10} = 0.3487$$

ii) Probability that all lines are busy

$$P(x = 10) = {}^{10}C_{10} (0.1)^{10} (0.9)^0 = (0.1)^{10}$$

iii) Probability that atleast one line is busy

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - 10c_0(0.1)^0(0.9)^{10}$$

$$P(x \geq 1) = 1 - 0.3487$$

$$P(x \geq 1) = \mathbf{0.6513}$$

iv) Probability that atmost 2 lines are busy

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$P(x \leq 2) = (0.9)^{10} + 10c_1(0.1)^1(0.9)^2 + 10c_2(0.1)^2(0.9)^8$$

$$P(x \leq 2) = \mathbf{0.9298}$$

6. In a quiz contest of answering 'Yes' or 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked ? Also find the probability of the same if there are 4 options for a correct answer.

Sol: Given $n = 10$

Let x denote the correct answer .

$$\text{i) } p = \frac{1}{2} = \mathbf{0.5}$$

$$\therefore q = 1 - p = 1 - 0.5$$

$$\mathbf{q = 0.5}$$

From binomial distribution,

$$P(x) = {}^nc_x p^x q^{n-x}$$

$$P(x) = 10c_x (0.5)^x (0.5)^{10-x}$$

$$\mathbf{P(x) = 10c_x (0.5)^{10}}$$

Probability of guessing atleast 6 answers correctly out of 10 questions

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$P(x \geq 6) = 10c_6(0.5)^{10} + 10c_7(0.5)^{10} + 10c_8(0.5)^{10} + 10c_9(0.5)^{10} + 10c_{10}(0.5)^{10}$$

$$P(x \geq 6) = (0.5)^{10} [10c_6 + 10c_7 + 10c_8 + 10c_9 + 10c_{10}]$$

$$P(x \geq 6) = (0.5)^{10} [210 + 120 + 45 + 10 + 1]$$

$$\mathbf{P(x \geq 6) = 0.3770}$$

$$\text{ii) } p = \frac{1}{4}$$

$$\therefore q = 1 - p = 1 - \frac{1}{4}$$

$$\mathbf{q = \frac{3}{4}}$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{10} C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$$

$$P(x) = {}^{10} C_x \frac{3^{10-x}}{4^{x+10-x}}$$

$$P(x) = {}^{10} C_x \frac{3^{10-x}}{4^{10}}$$

$$P(x) = \frac{1}{4^{10}} [3^{10-x} \cdot {}^{10} C_x]$$

Probability of guessing atleast 6 answers correctly out of 10 questions

$$P(x \geq 6) = P(x = 6) + P(x = 7) + P(x = 8) + P(x = 9) + P(x = 10)$$

$$P(x \geq 6) = \frac{1}{4^{10}} [3^4 \cdot {}^{10} C_6 + 3^3 \cdot {}^{10} C_7 + 3^2 \cdot {}^{10} C_8 + 3 \cdot {}^{10} C_9 + {}^{10} C_{10}]$$

$$P(x \geq 6) = \frac{1}{4^{10}} [81 \cdot 210 + 27 \cdot 120 + 9 \cdot 45 + 3 \cdot 10 + 1]$$

$$P(x \geq 6) = 0.0197$$

7. In sampling a large number of parts manufactured by a company, the mean number of defectives in samples of 20 is 2. Out of 1000 such samples how many would be expected to contain atleast 3 defective parts.

Sol: Given Mean, $\mu = 2$ and $n = 20$

$$np = 2$$

$$20p = 2$$

$$p = \frac{1}{10} = 0.1$$

$$q = 1 - p$$

$$q = 1 - \frac{1}{10} \quad q = \frac{9}{10} = 0.9$$

Let x denote the defective part.

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x}$$

$$P(x) = {}^{20} C_x (0.1)^x (0.9)^{20-x}$$

Probability of atleast 3 defective parts

$$P(x \geq 3) = 1 - P(x < 3)$$

$$P(x \geq 3) = 1 - [P(0) + P(1) + P(2)]$$

$$P(x \geq 3) = 1 - [20c_0(0.1)^0(0.9)^{20} + 20c_1(0.1)^1(0.9)^{19} + 20c_2(0.1)^2(0.9)^{18}]$$

$$P(x \geq 3) = 1 - 0.6769$$

$$P(x \geq 3) = 0.3231$$

Thus, the number of defectives in 1000 such samples = $1000 \times 0.3231 = 323.1 \approx 323$

8. If the mean and standard deviation of the number of correctly answered questions in a test given to 4096 students are 2.5 and $\sqrt{1.875}$. Find an estimate the number of candidates answering correctly i) 5 questions ii) 2 or less iii) 8 or more.

Sol: Given Mean = 2.5 $\sigma = \sqrt{1.875}$

Wkt, $\mu = np$ and $\sigma = \sqrt{V}$

Thus, $np = 2.5$ and $V = npq = 1.875$

$$\text{Now, } \frac{npq}{np} = \frac{1.875}{2.5}$$

We get $q = 0.75$, and $p = 1 - q = 0.25$

$$n = 10$$

Let x denote the number of correctly answered questions

$$P(x) = {}^nc_x p^x q^{n-x} = {}^{10}c_x (0.25)^x (0.75)^{10-x}$$

i) Probability that 5 questions are answered correctly

$$P(x = 5) = {}^{10}c_5 (0.25)^5 (0.75)^{10-5}$$

$$P(x = 5) = 0.0584$$

Estimation for 4096 students is $= 4096 \times 0.0584 \approx 239$

Number of students correctly answering 5 questions is **239**

ii) Probability that 2 or less questions are answered correctly

$$\begin{aligned} P(x \leq 2) &= P(x = 0) + P(x = 1) + P(x = 2) \\ &= {}^{10}c_0 (0.25)^0 (0.75)^{10} + {}^{10}c_1 (0.25)^1 (0.75)^9 + {}^{10}c_2 (0.25)^2 (0.75)^8 \end{aligned}$$

$$P(x \leq 2) = 0.5256$$

Estimation for 4096 students is $= 4096 \times 0.5256 \approx 2153$

Number of students correctly answering 2 or less than 2 questions is **2153**

iii) Probability that 8 or more questions are answered correctly

$$\begin{aligned} P(x \geq 8) &= P(x = 8) + P(x = 9) + P(x = 10) \\ &= {}^{10}c_8 (0.25)^8 (0.75)^2 + {}^{10}c_9 (0.25)^9 (0.75)^1 + {}^{10}c_{10} (0.25)^{10} (0.75)^0 \end{aligned}$$

$$P(x \geq 8) = 0.0005$$

Estimation for 4096 students is $=4096 \times 0.0005 \cong 2$

Number of students correctly answering 8 or more than 8 questions is 2

9. In 800 families with 5 children each how many families would be expected to have

i) 3 boys ii) 5 girls iii) either 2 or 3 boys iv) atmost 2 girls by assuming probabilities for boys and girls to be equal.

10. An airline knows that 5% of the people making reservations on a certain flight will not turn up. Consequently, their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will be a seat for every passenger who turns up?

11. 4 coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequencies.

Number of heads	0	1	2	3	4
Frequency	5	29	36	25	5

Sol: Let x denote the number of heads and f the corresponding frequency. Since the data is in the form of a frequency distribution we shall first calculate the mean.

$$\text{Mean, } \mu = \frac{\sum fx}{\sum f} = \frac{(0)(5) + (1)(29) + (2)(36) + (3)(25) + (4)(5)}{5 + 29 + 36 + 25 + 5} = \frac{196}{100} = 1.96$$

From binomial distribution, $\mu = np$. Here $n=4$

$$\text{Hence } 4p = 1.96$$

$$p = 0.49$$

$$q = 0.51$$

From binomial distribution,

$$P(x) = {}^n C_x p^x q^{n-x} \quad P(x) = {}^4 C_x (0.49)^x (0.51)^{4-x}$$

Since 4 coins were tossed, expected(theoretical) frequencies are obtained from

$$F(x) = 100P(x)$$

$$F(x) = 100 \cdot {}^4 C_x (0.49)^x (0.51)^{4-x} \quad \text{where } x = 0, 1, 2, 3, 4.$$

$$F(0) = 100 \cdot {}^4 C_0 (0.49)^0 (0.51)^4 = 6.765 \cong 7$$

$$F(1) = 100 \cdot {}^4 C_1 (0.49)^1 (0.51)^3 = 25.999 \cong 26$$

$$F(2) = 100 \cdot {}^4 C_2 (0.49)^2 (0.51)^2 = 37.47 \cong 37$$

$$F(3) = 100 \cdot {}^4 C_3 (0.49)^3 (0.51)^1 = 24.0004 \cong 24$$

$$F(4) = 100 \cdot {}^4 C_4 (0.49)^4 (0.51)^0 = 5.765 \cong 6$$

Thus the required theoretical frequencies are **7,26,37,24,6**.

Poisson Distribution

Poisson distribution is regarded as the limiting form of the binomial distribution when **n** is very large ($n \rightarrow \infty$) and **p** the probability of success is very small ($p \rightarrow 0$) so that **np** tends to a fixed finite constant say **m**.

Let **m** be a positive constant. Consider the infinite discrete probability distribution of a random variable **x** is given by $P(x) = \frac{m^x}{x!} e^{-m}$

The above probability density function is called the **poisson probability** function.

Mean and Standard deviation of the Poisson Distribution

$$\text{Mean, } \mu = m$$

$$\text{Variance, } V = m$$

$$\text{Std. Deviation, } \sigma = \sqrt{m}$$

Problems

1. The probabilities of a poisson variate taking the values 3 and 4 are equal. Find the PDF.

Also, calculate the probabilities of the variate taking the values 0 and 1.

Sol: Given $P(x = 3) = P(x = 4)$

$$\begin{aligned} \frac{m^3}{3!} e^{-m} &= \frac{m^4}{4!} e^{-m} \\ \frac{4!}{3!} &= \frac{m^4}{m^3} \end{aligned}$$

$$\therefore m = 4.$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m} = \frac{4^x}{x!} e^{-4}$$

$$P(x = 0) = \frac{4^0}{0!} e^{-4} = \mathbf{0.0183} \qquad P(x = 1) = \frac{4^1}{1!} e^{-4} = \mathbf{0.0733}$$

2. In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10. Use poisson distribution to calculate the approximate number of packets containing i) no defective ii) one defective iii) two defective blades in a consignment of 10,000 packets.

Sol: Probability of a defective blade = $p = \frac{1}{500} = 0.002$

In a packet of 10, the mean number of defective blades is

Mean, $m = np$

$$m = 10(0.002)$$

$$m = 0.02$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m}$$

$$P(x) = \frac{(0.02)^x}{x!} e^{-(0.02)}$$

$$P(x) = (0.9802) \frac{(0.02)^x}{x!} \quad (\text{Since } e^{-0.02} = 0.9802)$$

i) Probability of no defective

$$P(x = 0) = (0.9802) \frac{(0.02)^0}{0!} = 0.9802$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.9802 = 9802$$

ii) Probability of one defective

$$P(x = 1) = (0.9802) \frac{(0.02)^1}{1!} = 0.0196$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.0196 = 196$$

iii) Probability of one defective

$$P(x = 2) = (0.9802) \frac{(0.02)^2}{2!} = 0.0002$$

$$\text{For consignment of 10,000 packets} = 10,000 * 0.0002 = 2$$

3. The number of accidents in a year to taxi drivers in a city follows a Poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of the drivers with
i) no accident in a year ii) more than 3 accidents in a year.

Sol: Given Mean=3

Mean, $\mu = m$

$$m = 3$$

From poisson distribution,

$$P(x) = \frac{m^x}{x!} e^{-m}$$

$$P(x) = \frac{(3)^x}{x!} e^{-3}$$

$$P(x) = (0.05) \frac{(3)^x}{x!} \quad (\text{Since } e^{-3} = 0.05)$$

i) Probability of number of drivers with no accident in a year

$$P(x = 0) = (0.05) \frac{(3)^0}{0!} = 0.05$$

$$\text{Number of drivers out of 1000 with no accident in a year} = 1,000 * 0.05 = 50$$

ii) Probability of more than 3 accidents in a year

$$P(x > 3) = 1 - P(x \leq 3)$$

$$P(x > 3) = 1 - [P(0) + P(1) + P(2) + P(3)]$$

$$P(x > 3) = 1 - \left[(0.05) \frac{(3)^0}{0!} + (0.05) \frac{(3)^1}{1!} + (0.05) \frac{(3)^2}{2!} + (0.05) \frac{(3)^3}{3!} \right]$$

$$P(x > 3) = 1 - (0.05)[1 + 3 + 4.5 + 4.5]$$

$$P(x > 3) = 0.35$$

Number of drivers out of 1000 with more than 3 accidents in a year = 1,000 * 0.35 = 350

4. Given that 2% of the fuses manufactured by a firm are defective, find, by using poisson distribution, the probability that a box containing 200 fuses has i) no defective fuse ii) at least one defective fuse iii) exactly 3 defective fuses iv) 3 or more defective fuses.

Sol: The probability that a fuses is defective $p = 2/100 = 0.02$.

And $n = 200$

Mean, $\mu = np = 200 * 0.02$; **$m = 4$**

From poisson probability function $P(x) = \frac{m^x}{x!} e^{-m}$

$$P(x) = \frac{4^x}{x!} e^{-4}$$

Let x denotes the number of defective fuses in a box containing 200 fuses

i) Probability that the box contains no defective fuse

$$P(x = 0) = \frac{4^0}{0!} e^{-4} = \mathbf{0.0183}$$

ii) Probability that the box contains at least one defective fuse is

$$\begin{aligned} P(x \geq 1) &= 1 - P(x < 1) \\ &= 1 - P(x = 0) \\ &= 1 - 0.013 \end{aligned}$$

$$\mathbf{P(x \geq 1) = 0.9817}$$

iii) Probability that the box contains exactly 3 defective fuse

$$P(x = 3) = \frac{4^3}{3!} e^{-4}$$

$$\mathbf{P(x = 3) = 0.1952}$$

iv) Probability that the box contains 3 or more defective fuses

$$P(x \geq 3) = 1 - P(x < 3)$$

$$P(x \geq 3) = 1 - \left[\frac{4^0}{0!} e^{-4} + \frac{4^1}{1!} e^{-4} + \frac{4^2}{2!} e^{-4} \right]$$

$$P(x \geq 3) = 0.7621$$

5. A certain screw making machine has a chance of producing 2 defectives out of 1000. The screws are packed in boxes of 100. Using poisson distribution , find the approximate number of boxes containing i) no defective screw ii) one defective screw iii) two defective screw, in a consignment of 5000 boxes.

Sol: The probability that a screw is defective is given as $p = 2/1000 = 0.002$.

Here $n = 100$

Mean = $np = 100 \times 0.002 = 0.2$; **$m = 0.2$**

From poisson probability function $P(x) = \frac{m^x}{x!} e^{-m}$

$$P(x) = \frac{(0.2)^x}{x!} e^{-0.2}$$

Let x denotes the number of defective fuses in a box containing 100 screws

i) Probability that the box contains no defective screw

$$P(x = 0) = \frac{0.2^0}{0!} e^{-0.2} = 0.8187$$

Number of boxes containing no defective screw in 5000 boxes = $5000 \times 0.8187 = 4094$

ii) Probability that the box contains one defective screw is

$$P(x = 1) = \frac{(0.2)^1}{1!} e^{-0.2}$$

$$= 0.2 \times 0.8187$$

$$P(x = 1) = 0.1637$$

Number of boxes containing one defective screw in 5000 boxes = $5000 \times 0.1637 = 819$

iii) Probability that the box contains two defective screw is

$$P(x = 2) = \frac{(0.2)^2}{2!} e^{-0.2}$$

$$= 0.02 \times 0.8187$$

$$P(x = 2) = 0.0163$$

Number of boxes containing two defective screw in 5000 boxes = $5000 \times 0.0163 = 82$

6. A communication channel receives independent pulses at the rate of 12 pulses per micro second. The probability of transmission error is 0.001 for each micro second. Compute the probabilities of i) no error during a micro second ii) one error per micro second iii) atleast one per micro second iv) two errors v) atmost two errors.

Sol: Given $n = 12$ and $p = 0.001$

Mean number of errors in one micro second

$$\mu = np$$

$$m = 12 * 0.001$$

(Since $\mu = m$, from poisson distribution)

$$\mathbf{m = 0.012}$$

From Poisson distribution

$$P(x) = \frac{m^x}{x!} \cdot e^{-m}$$

$$P(x) = \frac{(0.012)^x}{x!} \cdot e^{-(0.012)}$$

$$P(x) = (0.9880) \frac{(0.012)^x}{x!} \quad (\text{Since } e^{-(0.012)} = 0.9880)$$

i) Probability of no error during a micro second

$$P(x = 0) = (0.9880) \frac{(0.012)^0}{0!}$$

$$\mathbf{P(x = 0) = 0.9880}$$

ii) Probability of one error during a micro second

$$P(x = 1) = (0.9880) \frac{(0.012)^1}{1!}$$

$$\mathbf{P(x = 1) = 0.0119}$$

iii) Probability of atleast one error during a micro second

$$P(x \geq 1) = 1 - P(x < 1)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - P(x = 0)$$

$$P(x \geq 1) = 1 - 0.9880$$

$$\mathbf{P(x \geq 1) = 0.012}$$

iv) Probability of two error during a micro second

$$P(x = 2) = (0.9880) \frac{(0.012)^2}{2!}$$

$$\mathbf{P(x = 2) = 0.00007}$$

v) Probability of atmost two error during a micro second

$$P(x \leq 2) = P(x = 0) + P(x = 1) + P(x = 2)$$

$$P(x \leq 2) = 0.9880 + 0.0119 + 0.00007$$

$$\mathbf{P(x \leq 2) = 0.99997}$$

Continuous Probability Distribution

If for every x belonging to the range of a continuous random variable X , we assign a real number $f(x)$ satisfying the conditions,

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)dx = 1$, then $f(x)$ is called a Continuous probability function or probability density function (p.d.f)

Cumulative distribution function

If X is a continuous random variable with probability density function $f(x)$ then the function $F(x)$ defined by $F(x) = P(X \leq x) = \int_{-\infty}^x f(x)dx$ is called the Cumulative distribution function (c.d.f) of X .

NOTE: If r is any real number, then

$$1. P(x \geq r) = \int_r^{\infty} f(x)dx$$

$$P(x < r) = 1 - P(x \geq r) = 1 - \int_r^{\infty} f(x)dx$$

Mean and Variance

If X is a continuous random variable with probability density function $f(x)$ where $-\infty < x < \infty$, the mean (μ) or expectation $E(X)$ and the variance (σ^2) of X is defined as follows.

$$\text{Mean, } \mu = \int_{-\infty}^{\infty} x \cdot f(x)dx$$

$$\text{Variance, } \sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx \quad \text{or} \quad V = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \mu^2$$

Problems

1. Find which of the following is a probability density function

$$a) f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad b) f_2(x) = \begin{cases} |x|, & 0 < |x| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$c) f_3(x) = \begin{cases} 2x, & 0 < x \leq 1 \\ 4 - 4x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Sol: Conditions for p.d.f are $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$

a) Clearly $f_1(x) \geq 0$

$$\int_{-\infty}^{\infty} f_1(x)dx = \int_0^1 f_1(x) dx = \int_0^1 2x dx = 2 \cdot \left[\frac{x^2}{2} \right]_0^1 = [1 - 0] = 1$$

$\therefore f_1(x)$ is a p.d.f

b) Evidently $f_2(x) = |x| \geq 0$

$$\int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^1 f_2(x) dx = \int_{-1}^1 |x| dx$$

$$\text{But, } |x| = \begin{cases} -x, & -1 < x < 0 \\ +x, & 0 < x < 1 \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^0 (-x) dx + \int_0^1 x dx$$

$$\int_{-\infty}^{\infty} f_2(x) dx = -\left[\frac{x^2}{2}\right]_{-1}^0 + \left[\frac{x^2}{2}\right]_0^1$$

$$\int_{-\infty}^{\infty} f_2(x) dx = -\left[0 - \frac{1}{2}\right] + \left[\frac{1}{2} - 0\right] = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore f_2(x)$ is a p.d.f

c) Given $f_3(x) = 2x > 0$ in $0 < x \leq 1$

But $f_3(x) = 4 - 4x$ is negative in $1 < x < 2$

The first condition is not satisfied.

$\therefore f_3(x)$ is not a p.d.f

2. Find the value of c such that $f(x) = \begin{cases} \frac{x}{6} + c, & 0 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$ is a p.d.f. Also find $P(1 \leq x \leq 2)$.

Sol: Here $f(x) \geq 0$ if $c \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\text{i.e., } \int_0^3 \left(\frac{x}{6} + c\right) dx = 1$$

$$\text{i.e., } \left[\frac{x^2}{12} + cx\right]_0^3 = 1$$

$$\text{i.e., } \frac{1}{12}(9 - 0) + c(3 - 0) = 1$$

$$\text{i.e., } \frac{9}{12} + 3c = 1$$

$$\text{i.e., } 3c = 1 - \frac{9}{12}$$

$$3c = \frac{3}{12}$$

$$\therefore c = \frac{1}{12}$$

$$\text{Now, } P(1 \leq x \leq 2) = \int_1^2 f(x) dx$$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$$

$$\begin{aligned}
&= \left[\frac{x^2}{12} + \frac{x}{12} \right]_1^2 \\
&= \frac{1}{12} (4 - 1) + \frac{1}{12} (2 - 1) \\
&= \frac{3}{12} + \frac{1}{12} = \frac{4}{12}
\end{aligned}$$

$$P(1 \leq x \leq 2) = \frac{1}{3}$$

3. Find the constant k such that $f(x) = \begin{cases} kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f. Also compute
i) $P(1 < x < 2)$ ii) $P(x \leq 1)$ iii) $P(x > 1)$ iv) Mean v) Variance

Sol: : Here $f(x) \geq 0$ if $k \geq 0$; Also we must have $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{i.e., } \int_0^3 kx^2 dx = 1$$

$$\text{i.e., } \left[\frac{kx^3}{3} \right]_0^3 = 1$$

$$\text{i.e., } \frac{k}{3} (27 - 0) = 1$$

$$\text{i.e., } 9k = 1$$

$$\text{i.e., } k = \frac{1}{9}$$

$$\therefore f(x) = \begin{cases} \frac{x^2}{9}, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\text{i) } P(1 < x < 2) &= \int_1^2 f(x)dx \\
&= \int_1^2 \frac{x^2}{9} dx \\
&= \left[\frac{x^3}{27} \right]_1^2 \\
&= \frac{1}{27} (8 - 1)
\end{aligned}$$

$$P(1 < x < 2) = \frac{7}{27}$$

$$\text{ii) } P(x \leq 1) = \int_0^1 f(x)dx$$

$$\begin{aligned}
P(x \leq 1) &= \int_0^1 \frac{x^2}{9} dx \\
&= \left[\frac{x^3}{27} \right]_0^1 \\
&= \frac{1}{27} (1 - 0)
\end{aligned}$$

$$P(x \leq 2) = \frac{1}{27}$$

$$\text{iii) } P(x > 1) = \int_1^3 f(x) dx$$

$$= \int_1^3 \frac{x^2}{9} dx$$

$$P(x > 1) = \left[\frac{x^3}{27} \right]_1^3$$

$$= \frac{1}{27} (27 - 1)$$

$$P(x > 1) = \frac{26}{27}$$

$$\text{iv) Mean, } \mu = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

$$= \int_0^3 x \cdot \frac{x^2}{9} dx$$

$$= \int_0^3 \frac{x^3}{9} dx$$

$$\mu = \left[\frac{x^4}{36} \right]_0^3$$

$$\mu = \frac{1}{36} (81 - 0)$$

$$\mu = \frac{81}{36}$$

$$\mu = \frac{9}{4}$$

$$\text{v) Variance, } V = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \mu^2$$

$$= \int_0^3 x^2 \cdot \frac{x^2}{9} dx - \left(\frac{9}{4} \right)^2$$

$$= \int_0^3 \frac{x^4}{9} dx - \frac{81}{16}$$

$$= \left[\frac{x^5}{45} \right]_0^3 - \frac{81}{16}$$

$$V = \frac{1}{45} [3^5 - 0] - \frac{81}{16}$$

$$V = \frac{243}{45} - \frac{81}{16}$$

$$V = \frac{81}{240}$$

$$V = \frac{27}{80}$$

Exponential Distribution

The continuous probability distribution having the probability density function $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}, \text{ where } \alpha > 0 \text{ is known as the } \mathbf{\text{exponential distribution}}.$$

Evidently $f(x) \geq 0$ and we have $\int_{-\infty}^{\infty} f(x)dx = 1$.

$f(x)$ satisfy both the conditions required for a continuous probability function.

Mean and standard deviation of the Exponential Distribution

$$\text{Mean, } \mu = \frac{1}{\alpha}$$

$$\text{Std. deviation, } \sigma = \frac{1}{\alpha}$$

Problems

1. If x is an exponential variate with mean 3 find i) $P(x > 1)$ ii) $P(x < 3)$

Sol: The p.d.f of the exponential distribution is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

The mean of the distribution is given by $\frac{1}{\alpha}$.

$$\text{By data, } \mu = \frac{1}{\alpha} = 3$$

$$\therefore \alpha = \frac{1}{3}$$

$$\text{Hence, } f(x) = \begin{cases} \frac{1}{3} e^{-\frac{x}{3}}, & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$\text{i) } P(x > 1) = 1 - P(x \leq 1)$$

$$P(x > 1) = 1 - P(x \leq 1)$$

$$= 1 - \int_0^1 f(x)dx$$

$$= 1 - \int_0^1 \frac{1}{3} e^{-\frac{x}{3}} dx$$

$$= 1 - \frac{1}{3} \int_0^1 e^{-\frac{x}{3}} dx$$

$$= 1 - \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^1$$

$$= 1 + \left[e^{-\frac{1}{3}} - e^{-0} \right]$$

$$= 1 + e^{-\frac{1}{3}} - 1$$

$$= e^{-\frac{1}{3}}$$

$$P(x > 1) = 0.7165$$

$$\begin{aligned} \text{ii) } P(x < 3) &= \int_0^3 f(x) dx \\ &= \int_0^3 \frac{1}{3} e^{-\frac{x}{3}} dx \\ &= \frac{1}{3} \int_0^3 e^{-\frac{x}{3}} dx \\ &= \frac{1}{3} \left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right]_0^3 \\ &= -[e^{-1} - e^{-0}] \\ &= -e^{-1} + 1 = 1 - \frac{1}{e} \end{aligned}$$

$$P(x < 3) = 0.6321$$

2. The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 minutes. Find the probability that a random call made from this booth i) ends less than 5 minutes ii) between 5 and 10 minutes.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 5$$

$$\therefore \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0 < x < \infty$$

$$\begin{aligned} \text{i) } P(x < 5) &= \int_0^5 f(x) dx \\ &= \int_0^5 \frac{1}{5} e^{-\frac{x}{5}} dx \\ &= \frac{1}{5} \int_0^5 e^{-\frac{x}{5}} dx \\ P(x < 5) &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_0^5 \\ &= -[e^{-1} - e^{-0}] \\ &= -e^{-1} + 1 = 1 - \frac{1}{e} \end{aligned}$$

$$P(x < 5) = 0.6321$$

$$\begin{aligned} \text{ii) } P(5 < x < 10) &= \int_5^{10} f(x) dx \\ &= \int_5^{10} \frac{1}{5} e^{-\frac{x}{5}} dx \\ &= \frac{1}{5} \int_5^{10} e^{-\frac{x}{5}} dx \end{aligned}$$

$$\begin{aligned} P(5 < x < 10) &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_5^{10} \\ &= -[e^{-2} - e^{-1}] \\ &= -e^{-2} + e^{-1} = \frac{1}{e} - \frac{1}{e^2} \end{aligned}$$

$$P(5 < x < 10) = 0.2325$$

3. In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for: (i) 10 minutes or more (ii) less than 10 minutes (iii) between 10 and 12 minutes.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 5$$

$$\therefore \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-\frac{x}{5}}, \quad 0 < x < \infty$$

$$\begin{aligned} \text{i) } P(x \geq 10) &= \int_{10}^{\infty} f(x) dx \\ &= \int_{10}^{\infty} \frac{1}{5} e^{-\frac{x}{5}} dx \\ &= \frac{1}{5} \int_{10}^{\infty} e^{-\frac{x}{5}} dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_{10}^{\infty} \\ &= -[e^{-\infty} - e^{-2}] \\ &= -e^{-\infty} + e^{-2} = -0 + e^{-2} \end{aligned}$$

$$P(x \geq 10) = 0.1353$$

$$\text{ii) } P(x < 10) = 1 - P(x \geq 10)$$

$$P(x < 10) = 1 - 0.1353$$

$$\mathbf{P(x < 10) = 0.8647}$$

$$\begin{aligned} \text{iii) } P(10 < x < 12) &= \int_{10}^{12} f(x) dx \\ &= \int_{10}^{12} \frac{1}{5} e^{-\frac{x}{5}} dx \\ &= \frac{1}{5} \int_{10}^{12} e^{-\frac{x}{5}} dx \\ &= \frac{1}{5} \left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}} \right]_{10}^{12} \\ &= -[e^{-2.4} - e^{-2}] \end{aligned}$$

$$P(10 < x < 12) = e^{-2} - e^{-2.4}$$

$$\mathbf{P(10 < x < 12) = 0.0446}$$

4. The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of a compressor of that company is i) less than 200 months ii) between 100 months and 25 years.

Sol: We have, $f(x) = \alpha e^{-\alpha x}$, $0 < x < \infty$

$$\text{By data, } \mu = \frac{1}{\alpha} = 200$$

$$\therefore \alpha = \frac{1}{200}$$

$$f(x) = \frac{1}{200} e^{-\frac{x}{200}}, \quad 0 < x < \infty$$

$$\begin{aligned} \text{i) } P(x < 200) &= \int_0^{200} f(x) dx \\ &= \int_0^{200} \frac{1}{200} e^{-\frac{x}{200}} dx \\ &= \frac{1}{200} \int_0^{200} e^{-\frac{x}{200}} dx \\ &= \frac{1}{200} \left[\frac{e^{-\frac{x}{200}}}{-\frac{1}{200}} \right]_0^{200} \\ &= -[e^{-1} - e^{-0}] \end{aligned}$$

$$= -e^{-1} + 1 = 1 - \frac{1}{e}$$

$$\mathbf{P(x < 200) = 0.6321}$$

Now, 25 years = 25 * 12 months = 300 months

$$\begin{aligned} \text{ii) } P(100 < x < 300) &= \int_{100}^{300} f(x) dx \\ &= \int_{100}^{300} \frac{1}{200} e^{-\frac{x}{200}} dx \\ &= \frac{1}{200} \int_{100}^{300} e^{-\frac{x}{200}} dx \\ &= \frac{1}{200} \left[\frac{e^{-\frac{x}{200}}}{-\frac{1}{200}} \right]_{100}^{300} \\ &= -[e^{-1.5} - e^{-2}] \end{aligned}$$

$$\mathbf{P(100 < x < 300) = 0.3834}$$

5. Determine k if random variable x has the density function $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$
hence find i) $P(x \geq 0)$ ii) $P(0 < x < 1)$ iii) c. d. f

Sol: Given f(x) is a probability density function.

$$\therefore f(x) \geq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\text{i.e., } \int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1$$

$$2k \int_0^{\infty} \frac{1}{1+x^2} dx = 1$$

$$2k[\tan^{-1}x]_0^{\infty} = 1 \quad ; \quad 2k[\tan^{-1}(\infty) - \tan^{-1}(0)] = 1$$

$$2k \left[\frac{\pi}{2} - 0 \right] = 1$$

$$k \cdot \pi = 1 \quad k = \frac{1}{\pi}$$

$$f(x) = \frac{\frac{1}{\pi}}{1+x^2}$$

$$\begin{aligned} \text{i) } P(x \geq 0) &= \int_0^{\infty} f(x) dx \\ &= \int_0^{\infty} \frac{\frac{1}{\pi}}{1+x^2} dx \end{aligned}$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \frac{1}{\pi} [\tan^{-1}x]_0^{\infty}$$

$$P(x \geq 0) = \frac{1}{\pi} [\tan^{-1}(\infty) - \tan^{-1}(0)]$$

$$P(x \geq 0) = \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right]$$

$$\mathbf{P(x \geq 0) = \frac{1}{2}}$$

$$\text{ii) } P(0 < x < 1) = \int_0^1 f(x) dx$$

$$= \int_0^1 \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_0^1 \frac{1}{1+x^2} dx$$

$$P(0 < x < 1) = \frac{1}{\pi} [\tan^{-1}x]_0^1$$

$$= \frac{1}{\pi} [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{4} - 0 \right]$$

$$\mathbf{P(0 < x < 1) = \frac{1}{4}}$$

$$\text{iii) c. d. } f = \int_{-\infty}^x f(x) dx$$

$$= \int_{-\infty}^x \frac{\frac{1}{\pi}}{1+x^2} dx$$

$$= \frac{1}{\pi} \int_{-\infty}^x \frac{1}{1+x^2} dx$$

$$\text{c. d. } f = \frac{1}{\pi} [\tan^{-1}x]_{-\infty}^x$$

$$= \frac{1}{\pi} [\tan^{-1}(x) - \tan^{-1}(-\infty)]$$

$$= \frac{1}{\pi} \left[\tan^{-1}x - \left(-\frac{\pi}{2} \right) \right]$$

$$\mathbf{c. d. } f = \frac{1}{\pi} \left[\tan^{-1}x + \frac{\pi}{2} \right] \quad \text{if } -\infty < x < \infty.$$

6. The daily turn over in a medical shop is exponentially distributed with Rs.6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds Rs.500 on a randomly chosen day.

Sol: Let 'x' denote the turn over per day.

∴ By exponential distribution

$$f(x) = \alpha e^{-\alpha x} ; x > 0.$$

Given, By data, $\mu = \frac{1}{\alpha} = 6000$

$$\therefore \alpha = \frac{1}{6000}$$

$$f(x) = \frac{1}{6000} e^{-\frac{x}{6000}}, \quad 0 < x < \infty$$

Let 'A' be the turn over for which the net profit is Rs.500.

Since the net profit is 8% of the turn over,

$$A * \frac{8}{100} = 500$$

$$\Rightarrow A = 6250$$

∴ The probability of profit exceeding Rs.500 is

$$\begin{aligned} P(x > 6250) &= \int_{6250}^{\infty} f(x) dx \\ P(x > 6250) &= \int_{6250}^{\infty} \frac{1}{6000} e^{-\frac{x}{6000}} dx \\ &= \frac{1}{6000} \int_{6250}^{\infty} e^{-\frac{x}{6000}} dx \\ &= \frac{1}{6000} \left[\frac{e^{-\frac{x}{6000}}}{-\frac{1}{6000}} \right]_{6250}^{\infty} \\ &= - \left[e^{-\infty} - e^{-\frac{6250}{6000}} \right] \\ &= e^{-\frac{6250}{6000}} \end{aligned}$$

$$P(x > 6250) = 0.3529$$

Normal Distribution

The continuous probability distribution having the probability density function $f(x)$ given

by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$ is known as the normal distribution.

Evidently $f(x) \geq 0$ and we have $\int_{-\infty}^{\infty} f(x)dx = 1$.

$f(x)$ satisfy both the conditions required for a continuous probability function.

Mean and standard deviation of the Normal Distribution

$$\text{Mean} = \mu$$

$$\text{Variance} = \sigma^2$$

Geometrically we can write

$$\text{i) } \int_{-\infty}^{\infty} \phi(z)dz = 1 \quad \text{ii) } \int_{-\infty}^0 \phi(z)dz = \int_0^{\infty} \phi(z)dz = \frac{1}{2}$$

Note

$$1. P(-\infty \leq z \leq \infty) = 1$$

$$2. P(-\infty \leq z \leq 0) = \frac{1}{2}$$

$$3. P(0 \leq z \leq \infty) \text{ or } P(z \geq 0) = \frac{1}{2}$$

$$\text{Also, } P(-\infty < z < z_1) = P(-\infty < z \leq 0) + P(0 \leq z < z_1)$$

$$\text{i.e., } P(z < z_1) = 0.5 + \phi(z_1)$$

$$\text{Also, } P(z > z_2) = P(z \geq 0) - P(0 \leq z < z_2)$$

$$\text{i.e., } P(z > z_2) = 0.5 - \phi(z_2)$$

Problems

1. Evaluate the following probabilities with the help of normal probability tables

i) $P(z \geq 0.85)$ ii) $P(-1.64 \leq z \leq -0.88)$ iii) $P(z \leq -2.43)$ iv) $P(|z| \leq 1.94)$

$$\text{Sol: i) } P(z \geq 0.85) = P(z \geq 0) - P(0 \leq z \leq 0.85)$$

$$= 0.5 - \phi(0.85)$$

$$= 0.5 - 0.3023$$

$$P(z \geq 0.85) = 0.1977$$

$$\text{ii) } P(-1.64 \leq z \leq -0.88)$$

$$\begin{aligned} \text{By Symmetry, } P(-1.64 \leq z \leq -0.88) &= P(0.88 \leq z \leq 1.64) \\ &= P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88) \\ &= \phi(1.64) - \phi(0.88) \\ &= 0.4495 - 0.3106 \end{aligned}$$

$$\mathbf{P(-1.64 \leq z \leq -0.88) = 0.1389}$$

$$\text{iii) } P(z \leq -2.43) = P(z \geq 2.43)$$

$$\begin{aligned} P(z \leq -2.43) &= P(z \geq 0) - P(0 \leq z \leq 2.43) \\ &= 0.5 - \phi(2.43) \\ &= 0.5 - 0.4925 \end{aligned}$$

$$\mathbf{P(z \geq -2.43) = 0.0075}$$

$$\text{iv) } P(|z| \leq 1.94) = P(-1.94 \leq z \leq 1.94)$$

$$\begin{aligned} &= 2 P(0 \leq z \leq 1.94) \\ &= 2 \cdot \phi(1.94) \\ &= 2 \cdot (0.4738) \end{aligned}$$

$$\mathbf{P(|z| \leq 1.94) = 0.9476}$$

2. If x is a normal variate with mean 30 and standard deviation 5 find the probability that
i) $26 \leq x \leq 40$ ii) $x \geq 45$

Sol: We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$ [By data $\mu = 30, \sigma = 5$]

i) To find $P(26 \leq x \leq 40)$

$$\text{If } x = 26, z = \frac{26-30}{5} = -0.8 \quad ; \quad \text{If } x = 40, z = \frac{40-30}{5} = 2$$

Hence we need to find, $\mathbf{P(-0.8 \leq z \leq 2)}$

$$\begin{aligned} P(-0.8 \leq z \leq 2) &= P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2) \\ &= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2) \\ &= \phi(0.8) + \phi(2) \\ &= 0.2881 + 0.4772 \end{aligned}$$

$$\mathbf{P(-0.8 \leq z \leq 2) = 0.7653}$$

ii) To find $P(x \geq 45)$

$$\text{If } x = 45, z = \frac{45-30}{5} = 3$$

Hence we need to find, $\mathbf{P(z \geq 3)}$

$$\begin{aligned}
P(z \geq 3) &= P(z \geq 0) - P(0 \leq z \leq 3) \\
&= 0.5 - \phi(3) \\
&= 0.5 - 0.4987
\end{aligned}$$

$$P(z \geq 3) = \mathbf{0.0013}$$

3. The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be
i) less than 65 ii) more than 75 iii) 65 to 75

Sol: Let x represent the marks of students

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$ [By data $\mu = 70, \sigma = 5$]

i) To find $P(x < 65)$

$$\text{If } x = 65, z = \frac{65-70}{5} = -1$$

Hence we need to find, $P(z < -1)$

$$\begin{aligned}
P(z < -1) &= P(z > 1) \\
&= P(z \geq 0) - P(0 \leq z \leq 1) \\
&= 0.5 - \phi(1) \\
&= 0.5 - 0.3413
\end{aligned}$$

$$P(z < -1) = \mathbf{0.1587}$$

Number of students scoring less than 65 marks = $1000 * 0.1587 = 158.7 \cong 159$

ii) To find $P(x > 75)$

$$\text{If } x = 75, z = \frac{75-70}{5} = 1$$

Hence we need to find, $P(z > 1)$

$$\begin{aligned}
P(z > 1) &= P(z \geq 0) - P(0 \leq z \leq 1) \\
&= 0.5 - \phi(1) \\
&= 0.5 - 0.3413
\end{aligned}$$

$$P(z > 1) = \mathbf{0.1587}$$

Number of students scoring more than 75 marks = $1000 * 0.1587 = 158.7 \cong 159$

iii) To find $P(65 < x < 75)$

Hence we need to find , $P(-1 < z < 1)$

$$P(-1 < z < 1) = 2P(0 < z < 1)$$

$$= 2\phi(1)$$

$$= 2(0.3413)$$

$$P(-1 < z < 1) = 0.6826$$

Number of students scoring marks between 65 to 75 marks = $1000 \times 0.6826 = 682.6 \cong 683$

4. In a test on electric bulbs, it was found that the life time of a particular brand was distributed normally with an average life of 2000 hours and S.D of 60 hours. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for i) more than 2100 hours ii) less than 1950 hours iii) between 1900 to 2100 hours.

Sol: Given $\mu = 2000$, $\sigma = 60$

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$

i) To find $P(x > 2100)$

$$\text{If } x = 2100, z = \frac{2100-2000}{60} = 1.67$$

Hence we need to find , $P(z > 1.67)$

$$P(z > 1.67) = P(z \geq 0) - P(0 < z < 1.67)$$

$$= 0.5 - \phi(1.67)$$

$$= 0.5 - 0.4525$$

$$P(z > 1.67) = 0.0475$$

$\therefore$ Number of bulbs that are likely to last for more than 2100 hours = 2500×0.0475

$$= 118.75 \cong 119$$

ii) To find $P(x < 1950)$

$$\text{If } x = 1950, z = \frac{1950-2000}{60} = -0.83$$

Hence we need to find , $P(z < -0.83)$

$$P(z < -0.83) = P(z > 0.83)$$

$$= P(z \geq 0) - P(0 < z < 0.83)$$

$$= 0.5 - \phi(0.83)$$

$$= 0.5 - 0.2967$$

$$P(z < -0.83) = 0.2033$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last for more than 1950 hours} &= 2500 * 0.2033 \\ &= 508.25 \cong 508\end{aligned}$$

iii) To find $P(1900 < x < 2100)$

$$\text{If } x = 1900, z = \frac{1900-2000}{60} = -1.67 \quad ; \text{ If } x = 2100, z = \frac{2100-2000}{60} = 1.67$$

Hence we need to find, $P(-1.67 < z < 1.67)$

$$\begin{aligned}P(-1.67 < z < 1.67) &= 2P(0 < z < 1.67) \\ &= 2 * \phi(1.67) \\ &= 2 * 0.4525\end{aligned}$$

$$P(-1.67 < z < 1.67) = 0.905$$

$$\begin{aligned}\therefore \text{Number of bulbs that are likely to last between 1900 and 2100 hours} \\ &= 2500 * 0.905 \\ &= 2262.5 \cong 2263\end{aligned}$$

5. In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find the mean and S.D of the distribution.

Sol: Let μ and σ be the mean and S.D of the normal distribution.

By data, $P(x < 45) = 0.31$ and $P(x > 64) = 0.08$

We have standard normal variate, $z = \frac{x-\mu}{\sigma}$

When $x = 45$, $z = \frac{45-\mu}{\sigma} = z_1$ (Say)

$x = 64$, $z = \frac{64-\mu}{\sigma} = z_2$ (Say)

So we have ,

$$P(z < z_1) = 0.31 \quad \text{and} \quad P(z > z_2) = 0.08$$

$$\text{i.e.,} \quad 0.5 + \phi(z_1) = 0.31 \quad \text{and} \quad 0.5 - \phi(z_2) = 0.08$$

$$\phi(z_1) = 0.31 - 0.5 \quad \text{and} \quad \phi(z_2) = 0.5 - 0.08$$

$$\phi(z_1) = -0.19 \quad \text{and} \quad \phi(z_2) = 0.42$$

Referring to the normal probability tables, we have

$$0.1915(\cong 0.19) = \phi(0.5) \quad \text{and} \quad 0.4192(\cong 0.42) = \phi(1.4)$$

$$\phi(z_1) = -\phi(0.5) \quad \text{and} \quad \phi(z_2) = \phi(1.4)$$

$$\begin{aligned} \Rightarrow \quad z_1 &= -0.5 & \text{and} & \quad z_2 = 1.4 \\ \text{i.e., } \frac{45-\mu}{\sigma} &= -0.5 & \text{and} & \quad \frac{64-\mu}{\sigma} = 1.4 \\ 45 - \mu &= -0.5\sigma & \text{and} & \quad 64 - \mu = 1.4\sigma \\ \mu - 0.5\sigma &= 45 & \text{and} & \quad \mu + 1.4\sigma = 64 \end{aligned}$$

Solving above equations, we get

$$\mu = 50 \quad \text{and} \quad \sigma = 10$$

Thus, Mean=50 and S.D=10

6. The mean weight of 500 students during a medical examination was found to be 50 kgs and S.D weight 6 kgs. Assuming that the weights are normally distributed, find the number of student having weight i) between 40 and 50 kgs ii) more than 60 kgs. [$\phi(1.67) = 0.4525$]

Sol: Given $\mu = 50$, $\sigma = 6$

We have standard normal variate, $z = \frac{x-\mu}{\sigma} = \frac{x-50}{6}$

i) **To find $P(40 < x < 50)$**

$$\text{If } x = 40, z = \frac{40-50}{6} = -1.67 \quad ; \text{ If } x = 50, z = \frac{50-50}{6} = 0$$

Hence we need to find, $P(-1.67 < z < 0)$

$$\begin{aligned} P(-1.67 < z < 0) &= P(0 < z < 1.67) \\ &= \phi(1.67) \end{aligned}$$

$$P(-1.67 < z < 0) = 0.4525$$

$$\begin{aligned} \therefore \text{Number of students having weight between 40 and 50 kgs} &= 500 * 0.4525 = 226.25 \\ &\cong 226 \end{aligned}$$

ii) **To find $P(x > 60)$**

$$\text{If } x = 60, z = \frac{60-50}{6} = 1.67$$

Hence we need to find, $P(z > 1.67)$

$$\begin{aligned} P(z > 1.67) &= P(z \geq 0) - P(0 < z < 1.67) \\ &= 0.5 - \phi(1.67) \\ &= 0.5 - 0.4525 \end{aligned}$$

$$P(z > 1.67) = 0.0475$$

$$\begin{aligned} \therefore \text{Number of students having weight more than 60 kgs} &= 500 * 0.0475 \\ &= 23.75 \cong 24 \end{aligned}$$

7. In an examination 7% of students score less than 35% marks and 89% of students score less than 60% of marks. Find the mean and S.D of the distribution if the marks are normally distributed. It is given that $P(1.2263)=0.39$ and $P(1.4757)=0.43$.

Sol: Let μ and σ be the mean and S.D of the normal distribution.

By data, $P(x < 35) = 0.07$ and $P(x < 60) = 0.89$

We have standard normal variate, $z = \frac{x-\mu}{\sigma}$

When $x = 35$, $z = \frac{35-\mu}{\sigma} = z_1$ (Say)

$x = 60$, $z = \frac{60-\mu}{\sigma} = z_2$ (Say)

So we have ,

$$P(z < z_1) = 0.07 \quad \text{and} \quad P(z < z_2) = 0.89$$

$$\text{i.e.,} \quad 0.5 + \phi(z_1) = 0.07 \quad \text{and} \quad 0.5 + \phi(z_2) = 0.89$$

$$\phi(z_1) = 0.07 - 0.5 \quad \text{and} \quad \phi(z_2) = 0.89 - 0.5$$

$$\phi(z_1) = -0.43 \quad \text{and} \quad \phi(z_2) = 0.39$$

Using the given data in the RHS of these, we have

$$\phi(z_1) = -\phi(1.4757) \quad \text{and} \quad \phi(z_2) = \phi(1.2263)$$

$$\Rightarrow \quad z_1 = -1.4757 \quad \text{and} \quad z_2 = 1.2263$$

$$\text{i.e.,} \quad \frac{35-\mu}{\sigma} = -1.4757 \quad \text{and} \quad \frac{60-\mu}{\sigma} = 1.2263$$

$$35 - \mu = -1.4757\sigma \quad \text{and} \quad 60 - \mu = 1.2263\sigma$$

$$\mu - 1.4757\sigma = 35 \quad \text{and} \quad \mu + 1.2263\sigma = 60$$

Solving above equations, we get

$$\mu = 48.65 \quad \text{and} \quad \sigma = 9.25$$

Thus, Mean = 48.65 and S.D = 9.25